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WIDE-BAND DIRECTIVITY OF
RECEIVING ARRAYS

BY

JAMES J. FARAN
ROBERT HILLS, JR.

MAY 1, 1953

ACOUSTICS RESEARCH LABORATORY
DIVISION OF APPLIED SCIENCE
HARVARD UNIVERSITY-CAMBRIDGE, MASSACHUSETTS

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Technical Memorandum 31

Wide-Band Directivity of Receiving Arrays

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Summary

The method of maximizing the directional gain of a receiving
array (heretofore useful only at a single frequency) is extended to
the case of operation at a finite bandwidth. It is also shown how to
design for maximum effective gain in the presence of noise which
might arise within the individual transducers or their preamplifiers.
Some necessary noise-field correlations are computed, and numerical
examples are included to show the effects of bandwidth and self-noise
on the overall gain for reception which can be achieved. The directional
gain of a broadside linear array for operation at a finite bandwidth is
always less than that for operation at a single frequency, but is always
greater, and often considerably greater, than the gain realized by use
of the single-frequency design at the finite bandwidth. Unlike the
single-frequency design, whose directional gain falls very rapidly as
the operating bandwidth is increased from zero, the wide-band designs
operate with good gain over a wide range of bandwidths.

Acoustics Research Laboratory

Division of Applied Science

Harvard University, Cambridge, Massachusetts

PREFACE

The subject of this memorandum is a by-product of a research project under way at this laboratory under the general heading "The Application of Correlation Techniques to Acoustic Receiving Systems." The principal results to this time have been reported in Technical Memoranda Nos. 27 and 28. The authors again wish to express their appreciation to Professor F. V. Hunt, who initiated this project, for his helpful and stimulating direction.

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Acoustics Research Laboratory

Harvard University, Cambridge, Massachusetts

I.

Introduction

Certain functions which appear in the usual formulation[1] of the problem
of maximizing the directional gain of an array can be considered to represent
simply special, single-frequency forms of a space correlation function which
can, in principle, be evaluated for any spectrum. Analysis of the directivity
problem from this point of view yields an extension of present design tech-
niques to the case of maximizing the directional gain of an array for wide-band
signal reception. It is becoming well known[2] that the computations involved
in maximizing the gain of an array for operation at a single frequency are very
laborious, and lead to specification of element currents, voltages, or sensi-
tivities which are very large compared to those for the uniform array and
whose magnitudes must be controlled with fantastic accuracy to achieve the
predicited directional gain. By the same token, the spacings and frequency
of operation must be accurately controlled to the same degree. The effect
of attempting to operate such an array with signals of finite bandwidth (in
transmission) or in the presence of a background noise of finite bandwidth
(in reception) is no less serious in reducing the directional gain. However,
the element sensitivities which maximize the directional gain of an array
used in reception can be determined in a straightforward manner for any
spectrum by inserting in the original formulation of the problem the proper
values of the space correlation function mentioned above. In an isotropic

[1] Pritchard, R. L., Directivity of Acoustic Linear Point Arrays, Technical
Memorandum No. 21, Acoustics Research Laboratory, Harvard University,
Cambridge, Massachusetts. (January 15, 1951).

[2] Yaru, N., "A Note on Super Gain Antenna Arrays", Proc. I. R. E. 39,
1081-1085 (September, 1951).

noise background (in terms of which the directional gain is defined) this
space correlation function can be readily computed for at least several
spectra, and is found to depend only on the distance between the elements
and the relative time delay of the signals after reception, if any. If these
background noise correlations can not be computed readily, and if they can be
measured, the design for maximum gain can be carried out in terms of
the measured values.

Because the reception problem depends almost entirely upon these
noise correlations, one may infer that the process of "shading" or selecting
these sensitivities for the greatest output signal-to-noise ratio is simply
one of cancelling, as much as possible, the coherent parts of the noises .
received by the various elements. If the spacing is so wide that there is
little coherence between the noises received by the various elements, there
is little that can be done to increase the directional gain by making the sensi-
tivities other than uniform. Because, in reception, the shading is an oper-
ation which is carried out only with regard to the background noise, the design
is not dependent upon the spectrum of the signal, as it is in the case of a
transmitting array, but only on the spectrum of the background noise at the
output of the receiver (assumed to contain no nonlinear circuits) whose
pass-band need only be as wide as is necessary to handle the signal.

It has not, perhaps, been generally recognized that the presence
of even very small amounts of self-noise in the individual elements or
preamplifiers of an array which has been designed for maximum gain can
very seriously decrease its effective gain. This is particularly true if any
of the sensitivities of the individual elements are considerably larger than
the over-all value for the uniform array. It will be demonstrated how such
noise, if it exists, can be taken into account in the design of the array, by
maximizing a suitably defined effective gain.

II.

Directional Gain and Effective Gain of an Array

Used in Reception

Assume that an array of m receiving transducers is located in a noisy

signal-bearing medium; that the signal and noise from each element is delayed a suitable time τ to compensate for the possibly different travel times from the signal source to the different elements, and adjusted in amplitude by the factor a, τ and a being, in general, different for each element; and that these signals are then added. Such an array is shown in Fig. 1. We assume, for convenience, that the mean-square amplitude of the signal at the output of each transducer is unity, and that the mean-square amplitude of the background noise at the output of each transducer is also unity. After the proper delay compensation for travel time, the signals from all elements will be in phase, and the total signal at the output will be

$$s(t) \sum_{i=1}^{m} a_i.$$

We apply the constraint

$$R = \sum_{i=1}^{m} a_i - 1 = 0 \tag{1}$$

to insure that, whatever the set of coefficients a_i, the mean-square signal amplitude at the output of the array will be unity. Now assume that the background noise received by the ith transducer is $n_i(t)$; after compensation and shading, the contribution of the ith transducer to the output noise is $a_i n_i(t - \tau_i)$, and the mean-square output noise from the entire array is

$$N^2 = \overline{\left[\sum_{i=1}^{m} a_i n_i(t - \tau_i) \right]^2}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \overline{n_i(t - \tau_i) n_j(t - \tau_j)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \rho_{i,j}(\tau_j - \tau_i) \tag{2}$$

where $\rho_{i,j}(\tau_j - \tau_i)$ is the cross-correlation function[3] of the background noises

[3] Faran, J. J., Jr., and R. Hills, Jr., Correlators for Signal Reception, Technical Memorandum No. 27, Acoustics Research Laboratory, Harvard University, Cambridge, Massachusetts. September 15, 1952, Chap. I.

received by the ith and j^{th} elements, and is normalized in the sense that $\rho_{i,i}(0) = \rho_{j,j}(0) = 1$, and also in the sense that

$$\rho_{i,j}(\tau_j - \tau_i) = \frac{\overline{n_i(t-\tau_i)n_j(t-\tau_j)}}{\sqrt{\overline{n_i^2(t)} \cdot \overline{n_j^2(t)}}} \,.$$

These correlation coefficients can be calculated easily for nondirectional transducers in an isotropic background noise field having certain spectra; these calculations are presented in the Appendix. For nonisotropic noise fields, if they could not readily be calculated, they might still be measured experimentally with sufficient accuracy to allow shading of an array for optimum operation in that particular nonisotropic background noise field. Because, by (1), the mean-square signal at the output of the array is unity, the mean-square noise output given by (2) is the reciprocal of the output signal-to-noise ratio, and because the signal-to-noise ratio at the input of any transducer was chosen to be unity, (2) is also the reciprocal of the signal-to-noise-improvement power ratio. In the special case where the correlation functions correspond to an isotropic noise background (defined in the Appendix), (2) is the reciprocal of the directivity factor, DF. However, because the analysis applies to cases where the background noise is not isotropic, if the correlation functions can be determined, we shall refer to (2) generally as the reciprocal of the signal-to-noise-improvement factor, IF, whence

$$\frac{1}{IF} = \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \rho_{i,j}(\tau_j - \tau_i). \tag{3}$$

If there is noise generated in the individual transducers (or their preamplifiers), the output signal-to-noise-improvement will be less than that given by (3), and can be considerably less, if the shading coefficients are large. In order to deal with possibility, we define the effective signal-to-noise-improvement factor EF to be the mean-square output signal-to-noise ratio when the mean-square input signal-to-noise ratio is unity. Obviously, if there is no transducer self-noise, EF = IF. Now let the rms amplitude of the self noise $n_i'(t)$ at the output of the ith receiver be β_i; then the mean-square noise at the output of the array (after compensation and

TRANSDUCERS

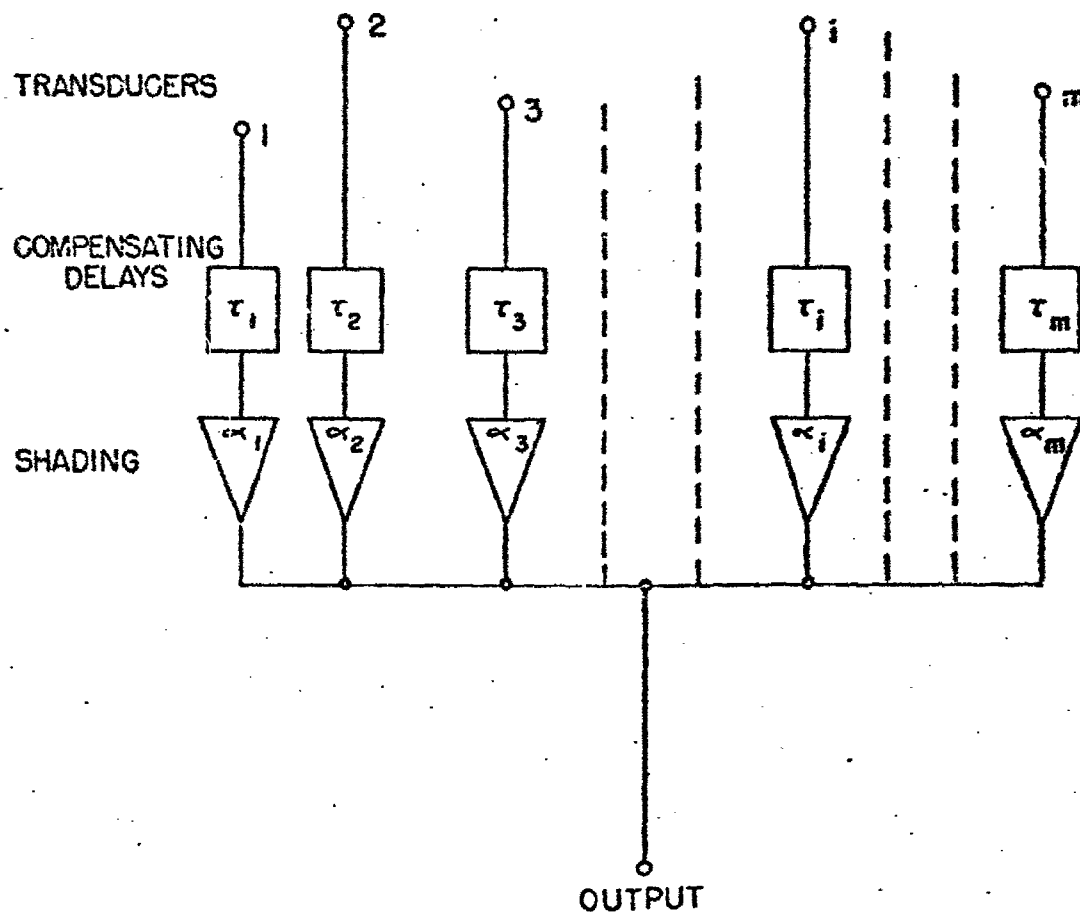COMPENSATING DELAYS

SHADING

OUTPUT

Fig. 1. Array of m receiving transducers with delay networks to compensate for the possibly different travel times from the signal source to the various elements and amplifiers whose gains are equal to the shading coefficients. The sum of the outputs of these amplifiers is the output of the array.

shading) is

$$N^2 \left(\sum_{i=1}^{m} a_i [n_i(t-\tau_i) + n_i'(t-\tau_i)] \right)^2$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \overline{n_i(t-\tau_i)n_j(t-\tau_j)} + \sum_{i=1}^{m} a_i^2 \overline{n_i'^2(t-\tau_i)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \rho_{i,j}(\tau_j - \tau_i) + \sum_{i=1}^{m} a_i^2 \beta_i^2 .$$

where it has been assumed that the self-noises are not coherent either with each other or with the noises from the medium. The effective signal-to-noise improvement factor is then given by

$$\frac{1}{EF} = \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \rho_{i,j}(\tau_j - \tau_i) + \sum_{i=1}^{m} a_i^2 \beta_i^2 . \tag{4}$$

When the correlation coefficients used in the above expressions correspond to an isotropic noise background (and there is no transducer self-noise), IF is equal to the directivity factor, and the directional gain of the array is simply this directivity factor expressed in decibels. In a similar manner (when there is self-noise), we can define an effective gain of the array to be equal to EF, as defined above and expressed in decibels, when the correlation coefficients correspond to an isotropic background.

III.

Maximization of the Directional Gain or of

the Effective Gain

The coefficients a_i for maximum DF, IF, or EF are readily determined by finding those values which produce extrema of the expressions (3) or (4) subject to the constraint (1). This is easily accomplished by the method of

the Lagrangian multiplier.[4] The coefficients are found by solving the following set of m+1 simultaneous linear equations in the m+1 unknowns $a_1, a_2, \ldots \ldots a_m$, and μ for the a's: For maximum improvement factor,

$$
\left.
\begin{aligned}
\frac{\partial(1/IF)}{\partial a_1} + \mu \frac{\partial R}{\partial a_1} &= 0 \\[2mm]
\frac{\partial(1/IF)}{\partial a_2} + \mu \frac{\partial R}{\partial a_2} &= 0 \\[2mm]
- - - - - - - - - - - - \\[2mm]
\frac{\partial(1/IF)}{\partial a_m} + \mu \frac{\partial R}{\partial a_m} &= 0
\end{aligned}
\right\} \qquad (5)
$$

$$ R = 0 \; ; $$

for maximum directivity factor DF or maximum effective improvement factor EF. IF in the equations above may be replaced by DF or EF, using the appropriate expression, (3) or (4).

These equations are considerably simplified in the case of a linear array of equally spaced elements arranged for broadside reception in an isotropic background, since no delays are needed and the shading will be symmetrical. We first renumber the elements to take advantage of this symmetry and reduce the number of unknowns. Whereas before, we numbered the m elements

$$1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots m,$$

we now number them

$$-(m-1)/2, \ldots\ldots\ldots -2, -1, 0, 1, 2, \ldots\ldots\ldots (m-1)/2,$$

if m is odd, and

$$-m/2, \ldots\ldots\ldots -2, -1, 1, 2, \ldots\ldots\ldots\ldots m/2\ldots$$

if m is even, where we now understand that $a_{-i} = a_i = a_{|i|}$. For nondirectional transducers in an isotropic background noise field, and with no artificial delays, the background noise crosscorrelations are a function only of the distance between the elements, as is shown in the Appendix. In general, it is convenient to write the crosscorrelation function as $P(d', \tau)$, where d' is the distance between the two points and τ is the relative (artificial) delay.

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[4]Sokolnikoff, I. S., and E. S. Sokolnikoff, Higher Mathematics for Engineers and Physicists, 2nd ed., McGraw-Hill, New York (1941), p. 163.

In terms of this function,

$$\rho_{i,j}(0) = P(|i-j|d, 0),$$

where d is the distance between adjacent elements $(d' = |i-j|d)$. Then, for m odd,

$$\frac{1}{DF} = \sum_{i=-(m-1)/2}^{(m-1)/2} \sum_{j=-(m-1)/2}^{(m-1)/2} a_{|i|} a_{|j|} P(|i-j|d, 0),$$

$$\frac{1}{EF} = \frac{1}{DF} + \sum_{i=-(m-1)/2}^{(m-1)/2} a_{|i|}^2 \beta_{|i|}^2 .$$

and

$$R = \sum_{i=-(m-1)/2}^{(m-1)/2} a_{|i|} - 1 = 0 .$$

$$\left.\right\} \quad (6)$$

while, for m even,

$$\frac{1}{DF} = \sum_{i=-m/2}^{m/2} \sum_{j=-m/2}^{m/2} a_{|i|} a_{|j|} P(|i-j|d, 0),$$

$$\frac{1}{EF} = \frac{1}{DF} + \sum_{i=-m/2}^{m/2} a_{|i|}^2 \beta_{|i|}^2 .$$

and

$$R = \sum_{i=-m/2}^{m/2} a_{|i|} - 1 = 0 .$$

$$\left.\right\} \quad (7)$$

Since, because of symmetry, differentiation with respect to a_{-i} gives the same equation as does differentiation with respect to a_i, it is only necessary to solve the following set of $(m+1)/2$ equations in the $(m+1)/2$ unknowns a_0, $a_1, \ldots \ldots a_{(m-1)/2}$, μ for the a's, if m is odd,

$$\frac{\partial(\frac{1}{DF})}{\partial a_0} + \mu \frac{\partial R}{\partial a_0} = 0$$

- - - - - - - - - - - - -

$$\left.\right] \quad (8)$$

$$\frac{\partial(\frac{1}{DF})}{\partial a_{(m-1)/2}} + \mu \frac{\partial R}{\partial a_{(m-1)/2}} = 0$$

$$R = 0;$$

or to solve the following set of $m/2+1$ equations for the a's, if m is even,

$$\frac{\partial(\frac{1}{DF})}{\partial a_1} + \mu \frac{\partial R}{\partial a_1} = 0$$

$$- - - - - - - - - - - - -$$

$$\frac{\partial(\frac{1}{DF})}{\partial a_{m/2}} + \mu \frac{\partial R}{\partial a_{m/2}} = 0$$

$$R = 0.$$

(9)

The directivity factor or effective signal-to-noise-ratio improvement factor achieved with a set of coefficients a_i is readily calculated by evaluating the appropriate expression in (6) or (7), whence the directional gain or effective gain is obtained by conversion to decibels.

It is perhaps well to bear in mind that the method of the Lagrangian multiplier only determines sets of coefficients which correspond to extrema of the function under consideration, and that whether this is a minimum (as we desire here, since we are working with the reciprocals of directivity factors) or a maximum must be determined afterwards. We have not, however, encountered anything other than minima in solving these equations.

IV.

Numerical Examples

The excitation coefficients and directivity indices, or directional gains, have been computed by this method for linear arrays of five and nine non-directional transducers. Calculations were made for a wide range of bandwidths of noise whose spectrum is the same as the response spectrum of a single series-tuned-circuit filter. The background noise in a receiving system would have this spectrum if the noise in the medium were essentially "white"

and if there were such a filter in the receiver. **The element-to-element** spacing of the arrays was 1/8 wavelength at the center frequency of the spectrum; the over-all lengths of the arrays were then 1/2 and 1 wavelength, respectively, at the center frequency. The shading coefficients for maximum directional gain at various bandwidths were found first by solving Eqs. (8) using the correlations of Table A-1; then the directional gains which resulted from using each particular set of shading coefficients at a number of different operating bandwidths were computed from the first of Eqs. (6).

The directional gain data for the five-element array are summarized in Table I. The first set of entries is for uniform shading where each a is equal to 1/5. The directional gain for single-frequency operation $(Q = \infty)$ is 1.62 db, and it increases somewhat as the bandwidth is increased. The next group of entries is for the case of shading to produce maximum directional gain for single-frequency operation. If the array could be used with a receiver that would accept only a single frequency, a directional gain of 5.54 db could be realized. If, however, the receiver pass-band was finite (as it must be in a practical case) but narrow, corresponding to a Q of 1000, for example, the directional gain would fail to -2.84 db, and it would continue to fall, and rapidly so, if the operating bandwidth were further increased. On the other hand, if the array were shaded for maximum directional gain at $Q = 1000$, a directional gain at this bandwidth of 3.91 db would result. As the rest of the data in this table indicate, a significant increase in the directional gain over that of the uniform array can be effected, but the increase is less at the wider bandwidths. Furthermore, the wider the design bandwidth, the wider the range of bandwidths at which the array will operate with a directional gain significantly greater than that for the uniform array. The data of this table are presented graphically in Fig. 2. Schematic diagrams of the shading coefficients which produce maximum directional gain at these different bandwidths are shown in Fig. 3; the wider the design bandwidth, the more nearly the shading coefficients approach the value for the uniform array. The increased weighting given the end-elements seems to be typical of wide-band maximum-gain designs.

The same data for the nine-element array are presented in Table II and

in graphical form in Fig. 4. Even starting with 13 significant figures, and carrying 12 through the computation, it was impossible to evaluate the coefficients with sufficient accuracy (8 significant figures) to compute the directional gain for this array for operation at a single frequency.[*] In order to make this example as complete as possible, a probable value for the directional gain for single-frequency operation, 7.6 db, has been estimated from Fig. 4-7 of Technical Memorandum No. 21.1 It was not possible to compute the directional gain for this shading at any other bandwidth, either, for the same reason. but the gain must fall very, very rapidly as the bandwidth is increased, if we are to judge by the performance of the five-element array. No difficulty was encountered in computing on a desk calculator the cases for the nine-element array where $Q=100$, 8, or 2, and these data show much the same behaviour as those for the smaller array. The shading coefficients are presented schematically for this case in Fig. 5. Here we see that by increasing the design bandwidth from zero to correspond to $Q=100$, the coefficients are made to fall from the order of 30,000 to the order of 1! The directional gain achieved at $Q=100$ is 1.42 db greater than that for the uniform array, and is unquestionably far greater than that for the single-frequency design, if it were operated at this bandwidth.

The effect of self-noise in the individual transducers (or their preamplifiers) on the effective gain of an array can be calculated from the second of Eqs. (6). For example, for the five-element linear array discussed above, designed for maximum directional gain for single-frequency operation, although the directional gain is 5.54 db, transducer self-noise 40 db below the noise from the medium at the output of each transducer will decrease the effective gain of the array to 0.15 db. If the self-noise is only 20 db below the noise from the medium, the effective gain of the array is reduced to -18.40 db! If, on the other hand, the array is designed for maximum effective gain in the presence of self-noise which is 20 db down, the effective gain can be made 3.63 db. These data are included with others for wide-band operation in Table III. Because

[*] It has been suggested that, once these coefficients are known approximately, by changing to a new set of variables scaled so that each of the new variables is of the order of unity, it might be possible to make the necessary improvement in the accuracy of the determined values. Unfortunately, time did not permit investigation of this technique.

Table I

DIRECTIONAL GAIN IN RECEPTION

for five-element linear array with spacing equal to 1/8
wavelength at center frequency of tuned-circuit spectrum

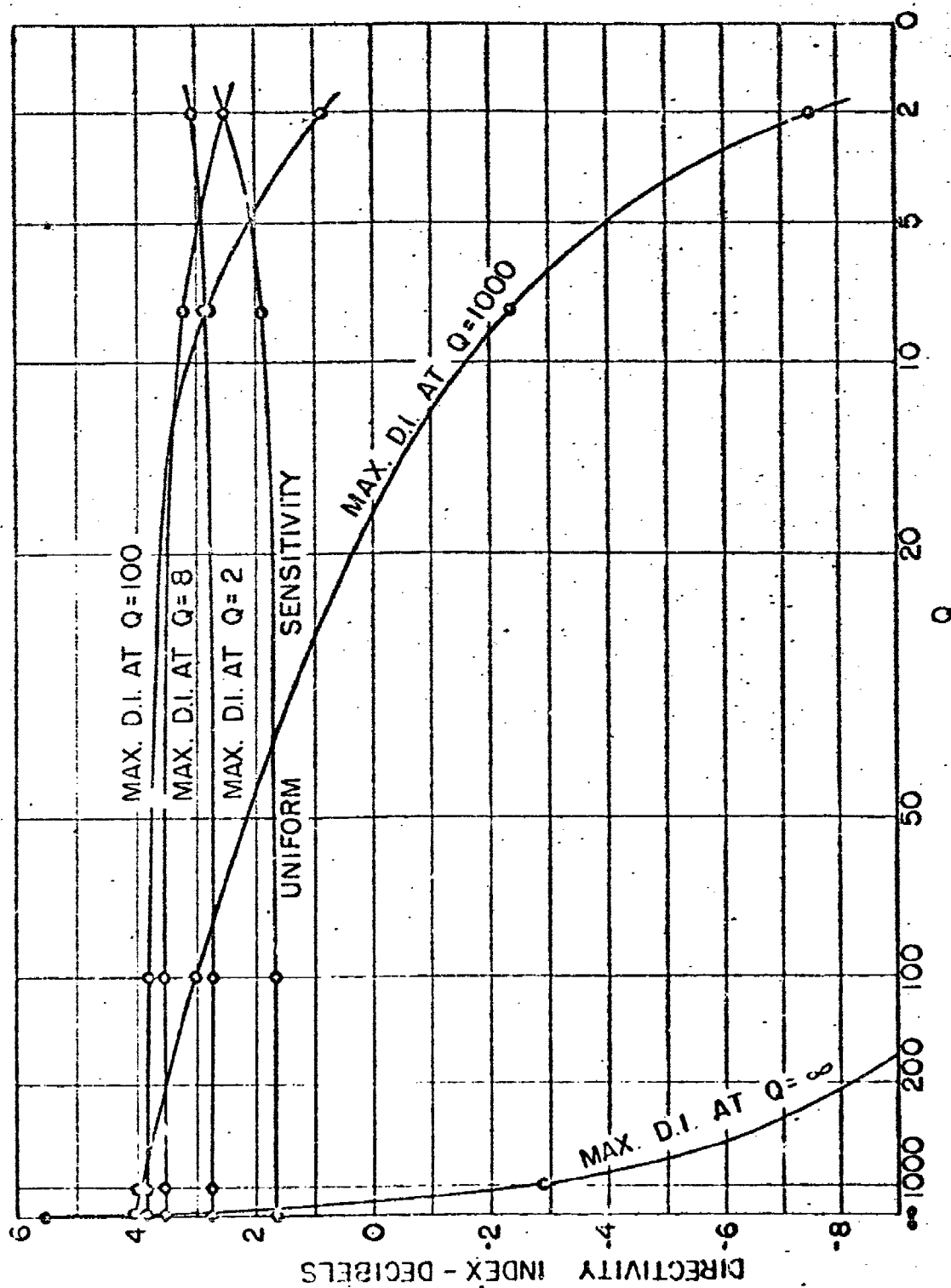| Shading	Q of Operating Spectrum	Directional Gain
Uniform	∞	1.62 db
"	8	1.84
"	2	2.47
Max. Gain at Q = ∞	∞	5.54
"	1000	-2.84
"	100	-12.22
"	8	-22.94
"	2	-29.12
Max. Gain at Q = 1000	∞	4.04
"	1000	3.91
"	100	2.99
"	8	-2.32
"	2	-7.50
Max. Gain at Q = 100	∞	3.83
"	1000	3.82
"	100	3.74
"	8	2.79
"	2	0.92
Max. Gain at Q = 8	∞	3.51
"	100	3.48
"	8	3.18
"	2	2.47
Max. Gain at Q = 2	∞	2.70
"	100	2.71
"	8	2.81
"	2	3.08

Fig. 2. Directivity index (directional gain) of a five-element linear array with spacing equal to 1/8 wavelength at the center frequency of a tuned-circuit spectrum as a function of the Q of the spectrum at which it is operated; for the uniformly sensitive array, and for arrays designed for maximum directivity index at the indicated values of the spectrum Q.
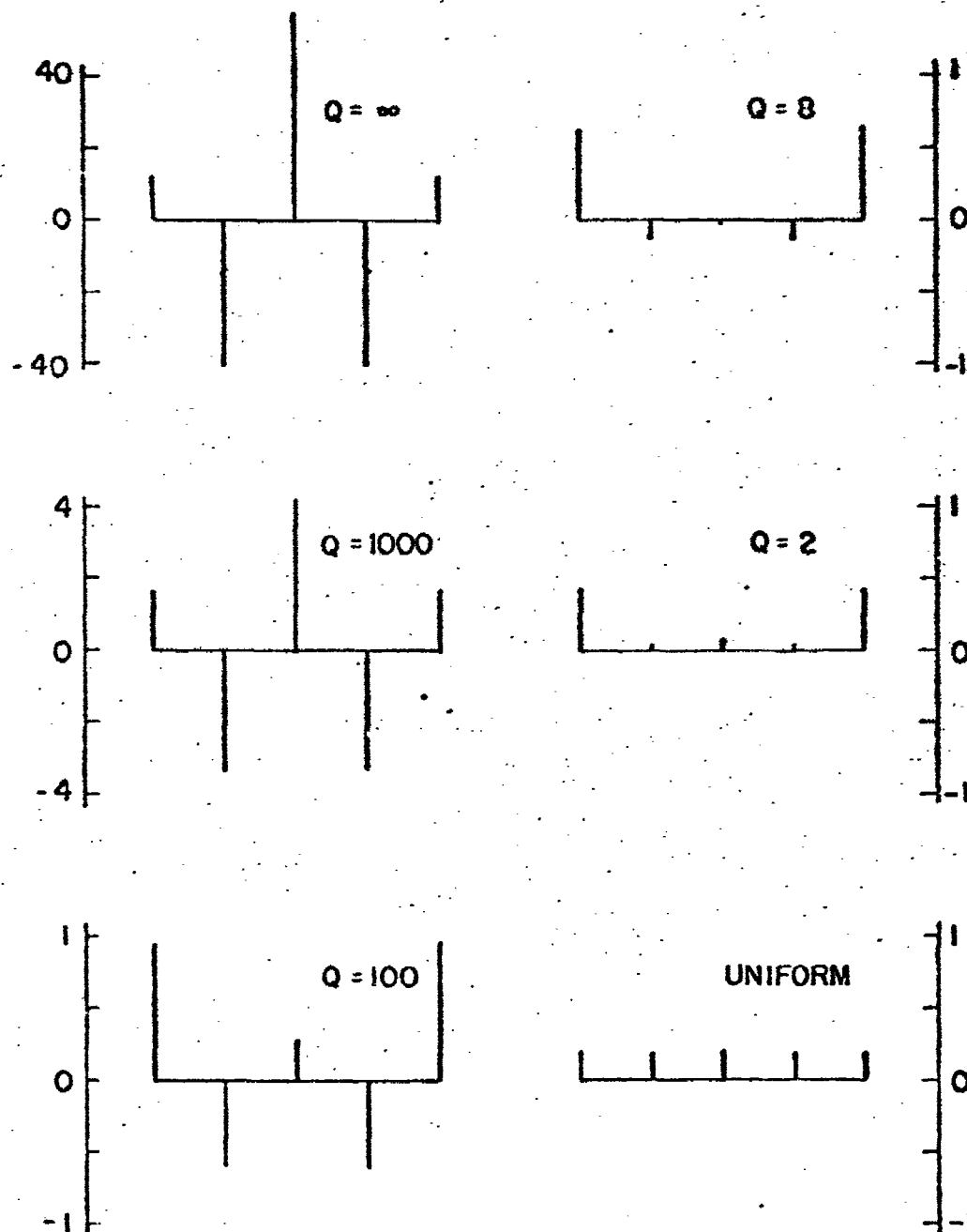
Fig. 3. Schematic diagrams of the shading coefficients for a
five-element linear array with spacing equal to 1/8 wavelength
at the center frequency of a tuned-circuit spectrum; for the
uniformly sensitive array, and for arrays designed for maximum
directivity index (directional gain) at the indicated value of the
spectrum Q.

Table II

DIRECTIONAL GAIN IN RECEPTION

for nine-element linear array with spacing equal to 1/8
wavelength at center frequency of tuned-circuit spectrum

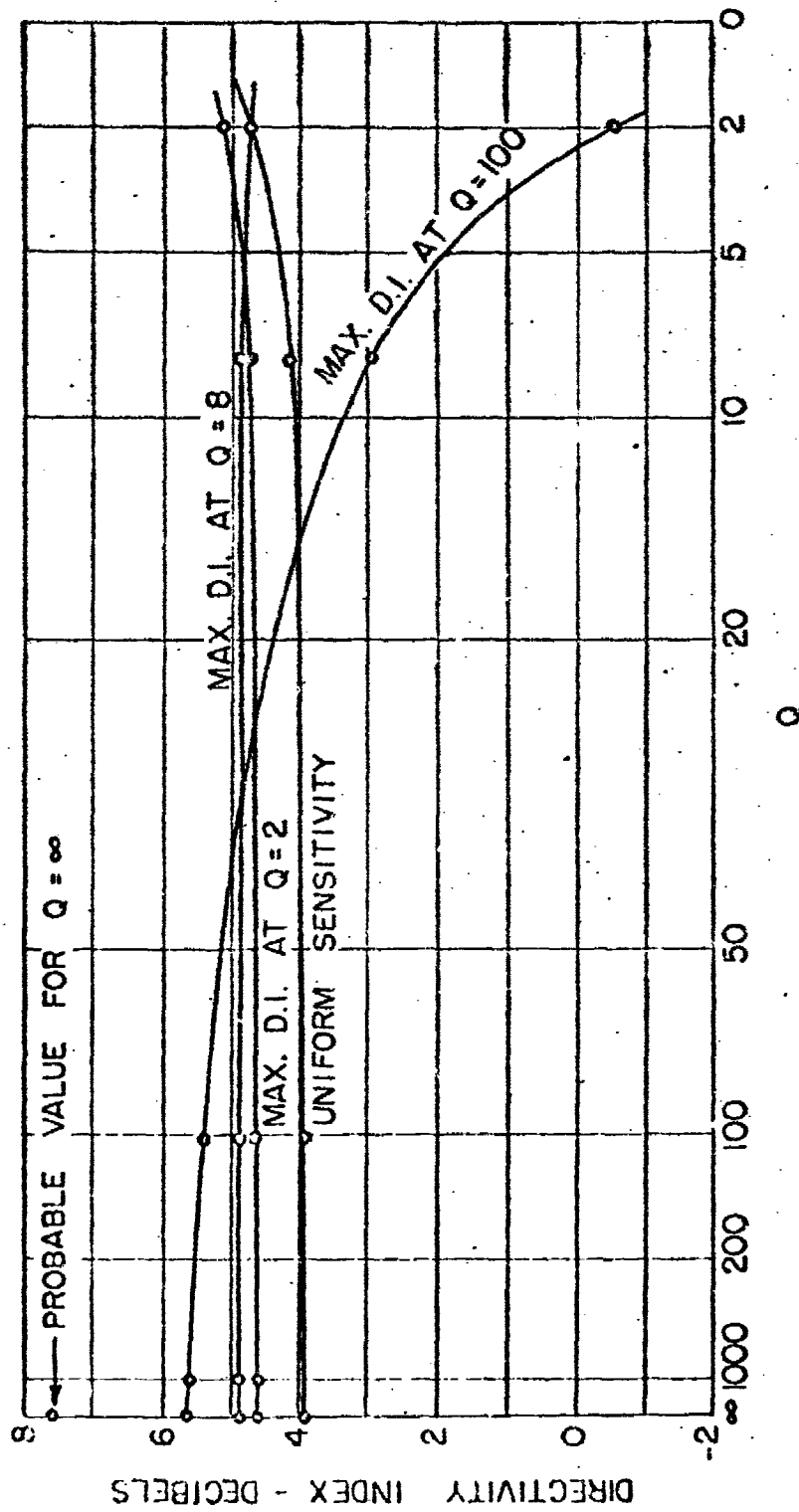| Shading	Q of Operating Spectrum	Directional Gain
Uniform	∞	3.94 db
"	100	3.96
"	8	4.12
"	2	4.66
Max. Gain at $Q=\infty$	∞	7.6 (estimated)
Max. Gain at $Q=100$	∞	5.68
"	1000	5.65
"	100	5.38
"	8	2.93
"	2	-0.56
Max. Gain at $Q=8$	∞	4.88
"	100	4.86
"	8	4.82
"	2	4.71
Max. Gain at $Q=2$	∞	4.61
"	1000	4.61
"	100	4.62
"	8	4.71
"	2	5.07

Fig. 4. Directivity index (directional gain) of a nine-element linear array with spacing equal to 1/8 wavelength at the center frequency of a tuned-circuit spectrum as a function of the Q of the spectrum at which it is operated; for the uniformly sensitive array, and for arrays designed for maximum directivity index at the indicated values of the spectrum Q.
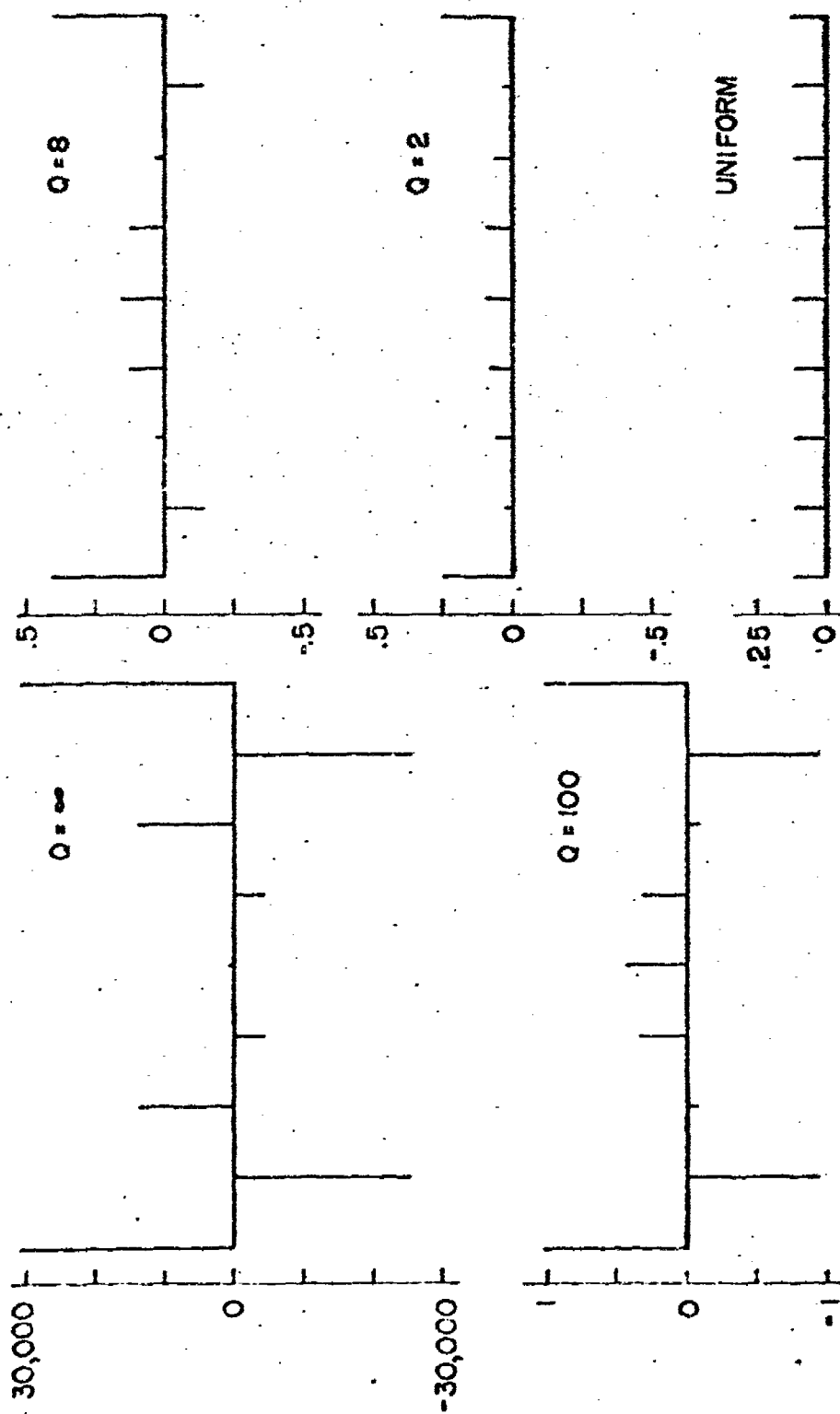
Fig. 5. Schematic diagrams of the shading coefficients for a nine-element, linear array with spacing equal to 1/8 wavelength at the center frequency of a tuned-circuit spectrum; for the uniformly sensitive array, and for arrays designed for maximum directivity index (directional gain) at the indicated value of the spectrum Q.

self-noise reduces the effective gain drastically only when the shading co-
efficients are very large, its effect is slight on arrays designed for wide-
band operation, where, as we see from Fig. 3, the shading coefficients are
relatively small. Included in Table III are the directional gains which these
self-noise designs would have if the self-noise were not present. These figures
indicate that it is possible to design so as to protect against the possible presence
of self-noise, while achieving a design which would still produce a fairly large
directional gain.

V.

Summary

A method has been demonstrated for determining the shading coefficients
of a receiving array which produces the greatest possible mean-square signal-
to-noise ratio at the output, in the presence of noise arising either in the signal-
bearing medium or in the transducers themselves or both. From the numerical
examples presented, it appears that the maximum directional gain of an equally-
spaced broadside array for wide-band operation is less than that for single-fre-
quency operation, but usually much more than that obtained by use of the single-
frequency design at the wide bandwidth, and, except for very wide-band cases,
is significantly greater than that obtained by using uniform sensitivity. The
effect of small amounts of self-noise arising in the transducers is small except
in arrays designed for maximum directional gain at very narrow bandwidths;
in these cases, by taking this self-noise into account, the effective gain of the
array can be made near to what the directional gain would be in the absence
of such self-noise.

Although it is now possible in principle to insert the functional form of the
noise crosscorrelation function into (3) or (4) and maximize the gain of the
array not only with respect to the shading coefficients but also with respect to
the spacings of the elements (along a line), his procedure has not been attempted
here, for fairly obvious reasons. By further accounting for the effects of
various time delays applied to the signals from the different elements, it might
be possible to maximize the gain of the array in terms of its spatial configu-
ration. Certainly only slight changes in the functions of Table A-1 result in
vastly different sets of coefficients for the equally spaced linear array; if some
of these slight changes can be compensated by changes in spacing along a line or

in spatial configuration of the array, increased directional gains at all band-
widths might be found.

Appendix

Computation of Background Noise Crosscorrelation in an

Isotropic Noise Field[*]

An isotropic noise field may be defined as one in which the total noise
power received by a directional receiver is independent of both its location
and its angular orientation. For example, imagine a uniform distribution
of infinitesimal statistically-independent random noise sources on the inner
surface of a very large sphere. If all these sources emit noise having the
same spectrum, the sound field in the vicinity of the center of the
sphere may be considered isotropic.[**] The crosscorrelation between the
total pressures at two different points in such an isotropic noise field can be
calculated by integrating the crosscorrelation between the pressures produced
at those two points by a single point source on the large sphere over all possible
positions of the point source on the sphere. This crosscorrelation has been
derived previously and numerically evaluated for the case of a rectangular
spectrum one octave in width.[5] We here again derive these results in somewhat
different notation, and also evaluate the crosscorrelation for tuned-circuit spectra.

Assume that a single point source is located on a large sphere concentric
with the origin at the spherical coordinate angles θ and ϕ. Assume that this
source generates random noise and that the pressure at the origin generated

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[*] The context of this appendix was previously presented as Chapter III of
Technical Memorandum No. 28, where use was made of it in a different
connection. Because of its fundamental importance to this work, it is
included here in substantially the same form, for the convenience of the
reader.

[**] The above definition is a broad-band equivalent of the background distribution
specified in the standard definition of directivity factor in American Standard
Acoustical Terminology, American Standards Association, Inc., New York
(July 31, 1951).

[5] Marsh, H. W., Jr., "Correlation in Wave Fields", A declassified portion of the
Quarterly Report for the period ending March 31, 1950, U. S. Navy Under-
water Sound Laboratory, New London, Connecticut.

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Table III

EFFECTIVE GAIN IN RECEPTION WITH NOISY TRANSDUCERS

for five-element linear array with spacing equal to 1/8 wavelength at center
frequency of tuned-circuit spectrum

Q of Spectrum	Self-Noise[*]	Designed for Maximum Directional Gain		Designed for Maximum Effective Gain	
		Directional Gain	Effective Gain	Directional Gain	Effective Gain
∞	-40 db	5.54 db	0.15 db		
"	-30	"	-8.56		
"	-20	"	-18.40	3.80 db	3.63 db
1000	-20	3.91	0.62	3.79	3.64
100	-20	3.74	3.47	3.71	3.57
8	-20	3.18	3.10	3.18	3.11
2	-20	3.08	3.05	3.08	3.05

[*] at output of each transducer in decibels relative to noise from medium at
output of that transducer

thereby may be represented by the random function f(t). The pressure at
a distance d from the origin in the positive direction on the axis of ϕ (z-axis)
is then $f[t+(d/c)\cos\theta]$, where c is the sound velocity in the medium. If $\rho(\tau)$
is the normalized autocorrelation function of f(t), the normalized cross-
correlation of the pressures at these two points is $\rho[\tau + (d/c)\cos\theta]$..

If the isotropic noise field is assumed to have a mean-square pressure
of unity in the vicinity of the origin, the contribution to this mean-square
pressure from an element of solid angle $\sin\theta\,d\theta\,d\phi$ will be $(1/4\pi)\sin\theta\,d\theta\,d\phi$.
The expression for the crosscorrelation in the isotropic noise field is then

$$P(d,\tau) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\rho\,[\tau+(d/c)\cos\theta]\,\sin\theta\,d\theta\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi}\rho\,[\tau+(d/c)\cos\theta]\,\sin\theta\,d\theta. \tag{A1}$$

The symbol capital rho (P) is used here to indicate that this crosscorrelation
function is normalized (in the sense that P(0,0) = 1). There are two possible
methods of evaluating this integral:

(1). We can introduce the new variable $\tau' = \tau+(d/c)\cos\theta$ so that

$$P(d,\tau) = \frac{c}{2d}\int_{-d/c+\tau}^{d/c+\tau}\rho(\tau')d\tau' \ .$$

This formula must be used with care whenever $\rho(\tau)$ is explicitly a function
of $|\tau|$; otherwise its use is convenient whenever the indefinite integral of
$\rho(\tau)$ is easily found.

(2). The autocorrelation function $\rho(\tau)$ may be written as the Fourier
transform of the intensity spectrum of f(t):[*]

- - - - - - -

[*]See reference 3. For convenience in carrying out integration in the
complex plane we assume here that $W(\omega)$ is an even function of ω
on the real axis, which is possible because $\rho(\tau)$ is real. A difference
of a factor 2 is implicit in the definitions of the spectra for positive,
real ω.

$$\rho(\tau) = \int_{-\infty}^{\infty} W(\omega) e^{j\omega\tau} d\omega.$$

Since $\rho(\tau)$ is a normalized autocorrelation function, the intensity spectrum $W(\omega)$ above is also assumed to be normalized so that

$$\int_{-\infty}^{\infty} W(\omega) d\omega = 1. \tag{A2}$$

Equation (A1) may then be written

$$P(d,\tau) = \frac{1}{2} \int_{0}^{\pi} \int_{-\infty}^{\infty} W(\omega) \, e^{j\omega[(d/c)\cos\theta + \tau]} \sin\theta \, d\omega \, d\theta$$

$$= \int_{-\infty}^{\infty} W(\omega) \, \frac{\sin(\omega d/c)}{(\omega d/c)} e^{j\omega\tau} d\omega. \tag{A3}$$

This formula is useful when $W(\omega)$ can be written explicitly as an analytic function of ω, in which case it may be readily evaluated by contour integration.

We first find $P(d, 0)$ for a rectangular spectrum. The intensity spectrum of the noise field is assumed to have the constant value $1/2\Delta\omega$ between $\omega_0 - \Delta\omega$ and $\omega_0 + \Delta\omega$, and to be zero elsewhere. From (A3) we then have

$$P(d, 0) = \frac{1}{2\Delta\omega} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} \frac{\sin(\omega d/c)}{(\omega d/c)} d\omega$$

$$= \frac{c}{2d\Delta\omega} \int_{0}^{\frac{d}{c}(\omega_0 + \Delta\omega)} \frac{\sin x}{x} dx - \frac{c}{2d\Delta\omega} \int_{0}^{\frac{d}{c}(\omega_0 - \Delta\omega)} \frac{\sin x}{x} dx$$

$$= \frac{c}{2d\Delta\omega} \left[Si(\frac{\omega_0 d}{c} + \frac{\Delta\omega d}{c}) - Si(\frac{\omega_0 d}{c} - \frac{\Delta\omega d}{c}) \right]$$

$$= \frac{c}{4\pi d\Delta f} \left[Si[\frac{2\pi d}{c}(f_0 + \Delta f)] - Si[\frac{2\pi d}{c}(f_0 - \Delta f)] \right],$$

where Si( ) is the sine-integral function.

Table A-1

VALUES OF THE FUNCTION P(d, 0) FOR THE TUNED-CIRCUIT SPECTRUM

as a function of d/λ_o, where λ_o is the wavelength at the
center frequency of the tuned-circuit spectrum, it having
been assumed that $\omega' = \omega_o$; for various values of Q.

d/λ_o	P(d, 0)		d/λ_o	P(d, 0)	
	$Q = \infty$			$Q = 8$	
0.0	1.0000	0000	0.0	1.0000	0000
0.125	0.9003	1632	0.125	0.8571	8930
0.25	0.6366	1977	0.25	0.5770	8975
0.375	0.3001	0544	0.375	0.2590	1131
0.5	0.0000	0000	0.5	0.0000	0000
0.625	-0.1800	6326	0.625	-0.1408	7477
0.75	-0.2122	0659	0.75	-0.1580	6968
0.875	-0.1286	1662	0.875	-0.0912	1545
1.0	0.0000	0000	1.0	0.0000	0000
	$Q = 1000$			$Q = 2$	
0.0	1.0000	0000	0.0	1.0000	0000
0.125	0.8999	6283	0.125	0.7398	1238
0.25	0.6361	1997	0.25	0.4298	6598
0.375	0.2997	5209	0.375	0.1665	1497
0.5	0.0000	0000	0.5	0.0000	0000
0.625	-0.1797	1005	0.625	-0.0674	6173
0.75	-0.2117	0718	0.75	-0.0653	3076
0.875	-0.1282	6355	0.875	-0.0325	3737
1.0	0.0000	0000	1.0	0.0000	0000
	$Q = 100$				
0.0	1.0000	0000			
0.125	0.8967	8772			
0.25	0.6316	3935			
0.375	0.2965	9065			
0.5	0.0000	0000			
0.625	-0.1765	6221			
0.75	-0.2072	6503			
0.875	-0.1251	2924			
1.0	0.0000	0000			

We now find the complete function $P(d,\tau)$ for the tuned-circuit spectrum. The intensity spectrum of "white" noise which has been passed through a series-tuned circuit is (normalized to satisfy (A2))

$$W(\omega) = \frac{(2\omega_F/\tau)\omega^2}{(\omega^2-\omega_0^2)^2 + 4\omega^2\omega_F^2} \; ,$$

where $\omega_0/2\pi$ is the center frequency, $\omega_F/2\pi$ is the half-band-width, and $Q = \omega_0/2\omega_F$. The function $W(\omega)$ is an analytic function of the complex variable ω having four simple poles, two each in the upper and lower half-planes. If the sine function in the integrand of (A3) is written as the sum of two exponentials, the integral may be evaluated by contour integration. On the real axis, $W(\omega)$ is a real, even function of ω, and $[\sin(\omega d/c)]/(\omega d/c)$ is also real and even. Because the Fourier transform of a real, even function is also a real, even function, it is only necessary to evaluate $P(d,\tau)$ for positive τ, the values for negative τ being obtained from

$$P(d, -\tau) = P(d,\tau).$$

We thus evaluate

$$P(d,\tau) = \frac{c}{2jd}\left[\int_{-\infty}^{\infty} \frac{W(\omega)}{\omega} e^{j\omega(d/c+\tau)}d\omega - \int_{-\infty}^{\infty} \frac{W(\omega)}{\omega} e^{-j\omega(d/c-\tau)}d\omega\right].$$

The contour integration is carried out in straightforward manner, assuming that $\tau \geq 0$, it being noted that the second integral is carried around the lower or upper half-plane depending respectively, on whether τ is less than, or greater than, d/c. The result is, for all τ,

$$P(d,\tau) = \frac{ce^{-\omega_F d/c}}{2d\omega'}\left[e^{-\omega_F|\tau|}\sin\omega'(\frac{d}{c}+|\tau|) + e^{\omega_F|\tau|}\sin\omega'(\frac{d}{c}-|\tau|)\right], \quad |\tau| \leq d/c,$$

$$= \frac{ce^{-\omega_F|\tau|}}{2d\omega'}\left[e^{-\omega_F d/c}\sin\omega'(\frac{d}{c}+|\tau|) + e^{\omega_F d/c}\sin\omega'(\frac{d}{c}-|\tau|)\right], |\tau| \geq d/c,$$

where

$$\omega' = \sqrt{\omega_0^2-\omega_F^2} = \omega_0\sqrt{1-/4Q^2}.$$

When $\tau = 0$ this crosscorrelation function reduces to

$$P(d, 0) = e^{-\omega_F d/c} \left[\frac{\sin(\omega' d/c)}{(\omega' d/c)} \right]. \qquad (A4)$$

Values of the function $P(d, 0)$ computed from (A4) for various values of Q are tabulated in Table A-I. In computing these values, the simplifying assumption that $\omega' = \omega_o$ was made. The numerical examples cited in this memorandum were based on the values given in this table; the result of having made the assumption is that the correct spacings of the arrays for which values were computed are larger than the value quoted by a factor

$$\frac{1}{\sqrt{1-1/4Q^2}}$$

are the true values of Q are smaller by a factor

$$\sqrt{1-1/4Q^2}$$

The effect on the computed directional gains of having made this assumption is almost certainly negligible.